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A NON-LINEAR MOVING MASS PROBLEM

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The longitudinal and transverse motions of a finite elastic beam traversed by a moving mass are presented. Using Hamilton's principle, two non-linear coupled differential equations governing the transverse and longitudinal displacements of the beam are developed. A finite difference method combined with a perturbation technique is used to solve the resulting boundary value problem. The results show the differences between the moving force and moving mass problems for the dynamic system. The effect of the friction force between the mass and beam on the longitudinal motion is shown to be significant.

1. INTRODUCTION

This paper deals with the longitudinal and transverse motions of a finite elastic beam traversed by a moving mass. Much exists in the literature on the subject, especially the transverse dynamics of the system, see references [1–7]. In that those works are linear, nothing exists on the longitudinal motions of such a system. Traditionally, there are two basic mathematical models which one can consider when simulating a bridge–vehicle interaction problem: the moving mass problem when the effect of the transverse inertia force between the beam and mass is considered and the moving force problem where the interactive inertial effect is neglected [6]. A difference in the two models is demonstrated in this paper.

A physical example of the system would be a locomotive traversing a bridge. In the design process one traditionally accounts for longitudinal motions of the bridge caused by thermal expansion. A question pursued here is the effect on longitudinal motions of the bridge caused by deformational and longitudinal dynamics. It will turn out that the friction force between the beam and moving mass has a significant effect on the longitudinal motion.

2. MATHEMATICAL MODEL

Consider a finite elastic beam (a bridge) traversed by a concentrated mass (a vehicle) as shown in Figure 1.

Let m be the mass of the beam per unit length and W the weight of the moving mass. Two non-linear coupled differential equations governing the transverse and longitudinal displacements of the system will be developed. To facilitate the boundary conditions (e.g., for a more general formulation of the boundary value problem) one uses Hamilton's principle to obtain the equations of motion and boundary conditions. A finite difference method combined with a perturbation technique will be used to solve the resulting boundary value problem. The algorithm developed here can easily be modified to solve

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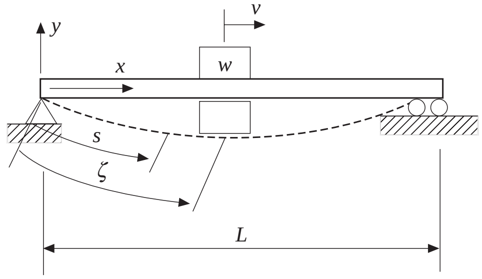

Figure 1. Elastic beam supporting a moving mass.

any conventional type of loading, such as a multi-wheeled vehicle (a train), a continuous moving mass (a tank), etc.

3. EQUATIONS OF MOTION

Hamilton's principle [8] is

$$\delta \int_{t_0}^{t_1} (T + U - V)\, \mathrm{d}t = 0. \tag{1}$$

The total potential energy of the deformed beam may be written as $V = V_1 + V_2$, where V_1 is the potential energy caused by bending and V_2 the potential energy caused by extension. Hence,

$$V_1 = \int_0^L \frac{1}{2} M\, \mathrm{d}\theta = \int_0^L \frac{EI}{2\rho}\frac{\mathrm{d}s}{\rho} = \int_0^L \frac{EI}{2}\frac{w_{ss}^2}{1 - w_s^2}\,\mathrm{d}s, \quad \text{and} \qquad V_2 = \int_0^L \frac{1}{2} A\sigma_x\varepsilon_x\, \mathrm{d}s, \tag{2}$$

where A is the cross-sectional area of the beam and w_s a derivative of the vertical component of displacement with respect to s. A list of symbols appears in the Appendix. From Figure 2 the strain in the horizontal direction may be expressed as

$$\varepsilon_x = \{[(dx + u_x dx)^2 + (w_x dx)^2]^{1/2} - dx\}/dx.$$

where u is the longitudinal displacement. Using a Taylor series expansion [10] one obtains

$$\varepsilon_x = u_x + \tfrac{1}{2}w_x^2 - \tfrac{1}{2}u_x w_x^2 + \tfrac{1}{2}u_x^2 w_x^2 - \tfrac{1}{8}w_x^4 + \cdots .$$

Note that u_x, w_x can be expressed as

$$u_x = u_s/(1 - w_s^2)^{1/2}, \qquad w_x = w_s/(1 - w_s^2)^{1/2}.$$

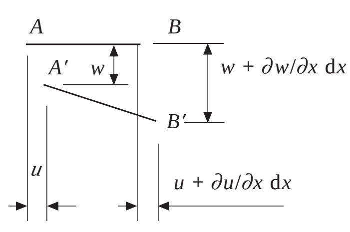

Figure 2. Displacement of beam element.

By Hooke's law and the chain rule V_2 can be written as a function of s as follows:

$$V_2 = \frac{1}{2}\int_0^L EA\left[\frac{u_s}{(1-w_s^2)^{1/2}} + \frac{1}{2}\frac{w_s^2}{(1-w_s^2)} - \frac{1}{2}\frac{u_s w_s^2}{(1-w_s^2)} + \frac{1}{2}\frac{u_s^2 w_s^2}{(1-w_s^2)^2}\right.$$

$$\left. - \frac{1}{8}\frac{w_s^4}{(1-w_s^2)^2} + \cdots \right]^2 \mathrm{d}s.$$

The kinetic energy of the beam is

$$T = \int_0^L \frac{m}{2}(\dot{w}^2 + \dot{u}^2)\,\mathrm{d}s,$$

and the work done by the external load is

$$U = \int_0^L (F_x u + F_y w)\,\mathrm{d}s + R_0^x u_0 - R_0^y w_0 + R_L^y w_L,$$

where an overdot represents a derivative with respect to time. From Figure 3 one sees that R_0^x, R_0^y and R_L^y are components of the reaction force at the supports, u_0, w_0 and w_L the displacements at the supports, and F_x and F_y the external concentrated force components of the moving mass. Utilising Hamilton's principle for the foregoing yields

$$\delta\int_{t_0}^{t_1}\left\{\int_0^L\left[\frac{EIw_{ss}^2}{2(1-w_s^2)} + \frac{EA}{2}f^2(u_s, w_s) - \frac{m}{2}\dot{w}^2 - \frac{m}{2}\dot{u}^2 - F_x u - F_y w\right]\mathrm{d}s\right.$$

$$\left. - R_0^x u_0 + R_0^y w_0 - R_L^y w_L\right\}\mathrm{d}t = 0,$$

where

$$f(u_s, w_s) = \frac{u_s}{(1-w_s^2)^{1/2}} + \frac{1}{2}\frac{w_s^2}{(1-w_s^2)} - \frac{1}{2}\frac{u_s w_s^2}{(1-w_s^2)^{3/2}} + \frac{1}{2}\frac{u_s^2 w_s^2}{(1-w_s^2)^2} - \frac{1}{8}\frac{w_s^4}{(1-w_s^2)^2} + \cdots.$$

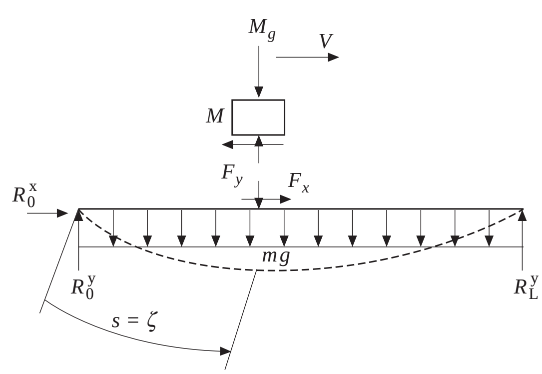

Figure 3. Free body diagram.

Noting that the denominators are of the form $[1 - w_s^2]^{-n/2}$, they are replaced by Taylor expansions

$$[1 - w_s^2]^{-n/2} = 1 + (n/2!)w_s^2 + (1/4!)6n(n/2 + 1)w_s^4 + \cdots,$$

where $n = 1, 2, 3, \ldots$.

Taking a variation of V_1, expressed by equation (2), and integrating by parts yields

$$\delta V_1 = EI\left\{\frac{w_{ss}}{1 - w_s^2}\,\delta w_s - \left[\frac{w_{sss}}{1 - w_s^2} + \frac{w_{ss}^2}{(1 - w_s^2)^2}\right]\delta w\right\}\Bigg|_0^L$$

$$+ EI\int_0^L \left\{\frac{w_{ssss}}{1 - w_s^2}\,\frac{4w_s w_{ss} w_{sss} + w_{ss}^3}{[1 - w_s^2]^2} + \frac{4w_s^2 w_{ss}^3}{[1 - w_s^2]^3}\right\}\delta w \,\mathrm{d}s.$$

Neglecting higher order terms $[O(u_s^5, w_s^5)]$, δV_1 reduces to

$$\delta V_1 = EI\{[w_{ss} + w_s^2 w_{ss}]\delta w_s - [w_s w_{ss}^2 + w_{sss} + w_s^2 w_{sss}]\delta w\}|_0^L.$$

$$+ EI\int_0^L [w_{ssss} + w_s^2 w_{ssss} + 4w_s w_{ss} w_{sss} + w_{ss}^3]\,\delta w\,\mathrm{d}s. \tag{3}$$

Similarly,

$$\delta V_2 = EA\int_0^L f(u_s, w_s)\,\delta[f(u_s, w_s)]\,\mathrm{d}s = EA\Big\{[f(u_s, w_s)f(u_s, w_s)_{u_s}]\delta u|_0^L$$

$$+ [f(u_s, w_s)f(u_s, w_s)_{w_s}]\delta w|_0^L - \int_0^L [f(u_s, w_s)f(u_s, w_s)_{u_s}]_s\delta u\,\mathrm{d}s$$

$$- \int_0^L [f(u_s, w_s)f(u_s, w_s)_{w_s}]_s\delta w\,\mathrm{d}s\Big\}.$$

Again, using a Taylor expansion and neglecting higher order terms $[O(u_s^5, w_s^5)]$, one obtains

$$\delta V_2 = EA\,\{[u_s + \tfrac{1}{2}w_s^2 + \tfrac{3}{2}u_s^2 w_s^2 + \tfrac{3}{8}w_s^4]\delta u\}|_0^L + EA\{[u_s w_s + u_s^3 w_s + \tfrac{1}{2}w_s^3 + \tfrac{3}{2}u_s w_s^3]\delta w\}|_0^L$$

$$- EA\int_0^L \{u_{ss} + 3u_s u_{ss} w_s^2 + w_s w_{ss} + 3u_s^2 w_s w_{ss} + \tfrac{3}{2}w_s^3 w_{ss}\}\delta u\,\mathrm{d}s$$

$$- EA\int_0^L \{u_{ss} w_s + 3u_s^2 u_{ss} w_s + \tfrac{3}{2}u_{ss} w_s^3 + u_s w_{ss} + u_s^3 w_{ss} + \tfrac{3}{2}w_s^2 w_{ss} + \tfrac{9}{2}u_s w_s^2 w_{ss}\}\delta w\,\mathrm{d}s. \tag{4}$$

The variation of the kinetic energy is

$$\delta \int_{t_0}^{t_1} T \,\mathrm{d}t = \delta \int_{t_0}^{t_1} \int_0^L \frac{m}{2} (\dot{w}^2 + \dot{u}^2) \,\mathrm{d}s \,\mathrm{d}t$$

$$= \int_0^L m(\dot{w}\delta w + \dot{u}\delta u)|_{t_0}^{t_1} \,\mathrm{d}s - \int_{t_0}^{t_1} \int_0^L m(\ddot{w}\delta w + \ddot{u}\delta u) \,\mathrm{d}s \,\mathrm{d}t \tag{5}$$

and the work done by external loads is

$$\delta \int_{t_0}^{t_1} U \,\mathrm{d}t = \delta \int_{t_0}^{t_1} \left\{ \int_0^L [F_x u + F_y w] \,\mathrm{d}s + R_0^x u_0 - R_0^y w_0 + R_L^y w_L \right\} \mathrm{d}t.$$

$$= \int_{t_0}^{t_1} \{ R_0^x \delta u_0 - R_0^y \delta w_0 + R_L^y \delta w_L \} \,\mathrm{d}t$$

$$+ \int_0^L \int_{t_0}^{t_1} (F_x \delta u + u \delta F_x + F_y \delta w + w \delta F_y) \,\mathrm{d}s \,\mathrm{d}t. \tag{6}$$

Now the external concentrated force components F_x and F_y, will be considered, which, from reference [6] may be expressed as

$$F_y(\zeta, t) = [W + (W/g)(\ddot{w} + 2v\dot{w}_s + v^2 w_{ss} + \dot{v} w_s)] \,\delta(s - \zeta) + mg, \tag{7}$$

$$F_x(\zeta, t) = \mu F_y(\zeta, t) = \mu[W + (W/g)(\ddot{w} + 2v\dot{w}_s + v^2 w_{ss} + \dot{v} w_s)] \,\delta(s - \zeta), \tag{8}$$

where $\delta(s - \zeta)$ is the Kronecker delta and μ the coefficient of friction between the vehicle and the bridge. Substituting equations (7) and (8) into equation (6) and integrating by parts yields

$$\delta \int_{t_0}^{t_1} U \,\mathrm{d}t = \int_{t_0}^{t_1} \int_0^L \{ \mu(W/g)(\ddot{u} + 2v\dot{u}_s + v^2 u_{ss} + \dot{v} u_s) \,\delta(s - \zeta)$$

$$+ [W + 2(W/g)(\ddot{w} + 2v\dot{w}_s + v^2 w_{ss} + \dot{v} w_s)] \,\delta(s - \zeta) + mg \} \,\delta w \,\mathrm{d}s \,\mathrm{d}t$$

$$+ \int_{t_0}^{t_1} \int_0^L \mu[W + (W/g)(\ddot{w} + v\dot{w}_s + v^2 u_{ss} + \dot{v} w_s)] \,\delta(s - \zeta) \,\delta u \,\mathrm{d}s \,\mathrm{d}t$$

$$+ \int_{t_0}^{t_1} \mu(Wv^2/g) u \,\delta(s - \zeta) \delta w_s|_0^L \,\mathrm{d}t - \int_{t_0}^{t_1} [2\mu(Wv/g)\dot{u}$$

$$+ \mu\,(Wv^2/g) u_s] \,\delta(s - \zeta) \delta w|_0^L \,\mathrm{d}t + \int_{t_0}^{t_1} (Wv^2/g) w \,\delta(s - \zeta) \delta w_s|_0^L \,\mathrm{d}t$$

$$-\int_{t_0}^{t_1}[2(Wv/g)\dot{w}+(Wv^2/g)w_s]\,\delta(s-\zeta)\delta w|_0^L\,\mathrm{d}t$$

$$+\int_{t_0}^{t_1}(R_0^x u_0-R_0^y w_0+R_L^y w_L)\,\mathrm{d}t. \tag{9}$$

Substituting equations (3–5) and (9) into equation (1) and assuming the velocity v to be constant, the governing equations of motion and boundary conditions of the system, to order $[o(u_s^5, w_s^5)]$, can be written as

$$\begin{aligned}&EA[u_{ss}+3u_s u_{ss}w_s^2+w_s w_{ss}+3u_s^2 w_s w_{ss}+\tfrac{3}{2}w_s^3 w_{ss}]-m\ddot{u}\\&\quad=-\mu[W+(W/g)(\ddot{w}+2v\dot{w}_s+v^2 w_{ss})]\,\delta(s-\zeta),\end{aligned} \tag{10}$$

$$\begin{aligned}&EI[w_{ssss}+w_s^2 w_{ssss}+4w_s w_{ss}w_{sss}+w_{ss}^3]-EA[u_{ss}w_s+3u_s^2 u_{ss}w_s+\tfrac{3}{2}u_{ss}w_s^3\\&\quad+u_s w_{ss}+u_s^3 w_{ss}+\tfrac{3}{2}w_s^2 w_{ss}+\tfrac{9}{2}w_s w_s^2 w_{ss}]+m\ddot{w}\\&=mg+[W+2(W/g)(\ddot{w}+2v\dot{w}_s+v^2 w_{ss})]\,\delta(s-\zeta)\\&\quad+\mu(W/g)(\ddot{u}+2v\dot{u}_s+v^2 u_{ss})\,\delta(s-\zeta).\end{aligned} \tag{11}$$

For the illustrative problems contained herein a simply supported beam will be considered. The boundary conditions reduce to

$$w_{ss}|_0^L=0,\qquad w|_0^L=0,\qquad u_0=0,\qquad [u_s+\tfrac{1}{2}w_s^2+\tfrac{3}{2}u_s^2 w_s^2+\tfrac{3}{8}w_s^4)]|_{s=L}=0. \tag{12}$$

To non-dimensionalize the system one now defines the following non-dimensional quantities

$$\hat{s}=s/L,\qquad \hat{w}=w/L,\qquad \hat{u}=u/L,\qquad \hat{\zeta}=\zeta/L=vt/L. \tag{13}$$

Making the appropriate substitutions of equations (13) into equations (10–12) yields the equations of motion in non-dimensional form:

$$\begin{aligned}&\hat{u}_{\hat{s}\hat{s}}+3\hat{u}_{\hat{s}}\hat{u}_{\hat{s}\hat{s}}\hat{w}_{\hat{s}}^2+\hat{w}_{\hat{s}}\hat{w}_{\hat{s}\hat{s}}+3\hat{u}_{\hat{s}}^2\hat{w}_{\hat{s}}\hat{w}_{\hat{s}\hat{s}}+\tfrac{3}{2}\hat{w}_{\hat{s}}^3\hat{w}_{\hat{s}\hat{s}}-(1/\omega_1^2)\ddot{\hat{u}}+\mu[\beta_7+\beta_1^2(1/\omega_2^2)\ddot{\hat{w}}\\&\qquad+(2/\omega_2)\dot{\hat{w}}_s+\hat{w}_{\hat{s}\hat{s}})]\,\delta(\hat{s}-\hat{\zeta})=0\end{aligned} \tag{14}$$

and

$$\begin{aligned}&\hat{w}_{\hat{s}\hat{s}\hat{s}\hat{s}}+\hat{w}_{\hat{s}}^2\hat{w}_{\hat{s}\hat{s}\hat{s}\hat{s}}+4\hat{w}_{\hat{s}}\hat{w}_{\hat{s}\hat{s}}\hat{w}_{\hat{s}\hat{s}\hat{s}}+\hat{w}_{\hat{s}\hat{s}}^3-\beta_4[\hat{u}_{\hat{s}\hat{s}}\hat{w}_{\hat{s}}+3\hat{u}_{\hat{s}}^2\hat{u}_{\hat{s}\hat{s}}\hat{w}_{\hat{s}}+\tfrac{3}{2}\hat{u}_{\hat{s}\hat{s}}\hat{w}_{\hat{s}}^3+\hat{u}_{\hat{s}}\hat{w}_{\hat{s}\hat{s}}\\&\quad+\hat{u}_{\hat{s}}^3\hat{w}_{\hat{s}\hat{s}}+\tfrac{3}{2}\hat{w}_{\hat{s}}^2\hat{w}_{\hat{s}\hat{s}}+\tfrac{9}{2}\hat{u}_{\hat{s}}\hat{w}_{\hat{s}}^2\hat{w}_{\hat{s}\hat{s}}]-2\beta_2((1/\omega_2^2)\ddot{\hat{w}}+(2/\omega_2)\dot{\hat{w}}_{\hat{s}}+\hat{w}_{\hat{s}\hat{s}})\,\delta(\hat{s}-\hat{\zeta})\\&\quad-\mu\beta_2(1/\omega_2^2)\ddot{\hat{u}}+(2/\omega_2)\dot{\hat{u}}_{\hat{s}}+\hat{u}_{\hat{s}\hat{s}})\,\delta(\hat{s}-\hat{\zeta})+(1/\beta_5\omega_3^2)\ddot{\hat{w}}-\beta_6=0,\end{aligned} \tag{15}$$

where

$$\omega_1=(1/L)\sqrt{EA/m},\qquad \omega_2=v/L,\qquad \omega_3=(1/L^2)\sqrt{gLEI/W},\qquad \omega_4=(1/L)\sqrt{gLEA/W}$$

$$\beta_1=v/\omega_4 L,\qquad \beta_2=v/\omega_3 L,\qquad \beta_4=AL^2/I,\qquad \beta_5=W/mgL,$$

$$\beta_6=(g/\omega_3^2 L)[\delta(\hat{s}-\hat{\zeta})+1/\beta_5],\qquad \beta_7=W/EA.$$

Similarly the boundary conditions become

$$\hat{w}_{\hat{s}\hat{s}}|_0^L=0,\qquad \hat{w}|_0^L=0,\qquad \hat{u}|_{\hat{s}=0}=0,\qquad (\hat{u}_{\hat{s}}+\tfrac{1}{2}\hat{w}_{\hat{s}}^2+\tfrac{3}{2}\hat{u}_{\hat{s}}^2\hat{w}_{\hat{s}}^2+\tfrac{3}{8}\hat{w}_{\hat{s}}^4)|_{\hat{s}=1}=0,$$

with initial conditions

$$\hat{w} = \dot{\hat{w}} = \hat{u} = \dot{\hat{u}} = 0, \qquad \text{at} \quad t = 0.$$

4. FINITE DIFFERENCE EQUATIONS

To obtain a solution to equations (14) and (15) an implicit method was used. First the system was linearized and transformed to a system of difference equations in matrix form. The solution to the linear system was used as a first approximation to determine the solution to the non-linear equations. A perturbation between the linear and non-linear systems was then employed. This was continued until convergence was achieved.

Specifically one begins by using the Crank–Nicolson method [9] (replacing each term by the mean of its finite difference representation at the $(j+1)$th, jth and $(j-1)$th times) to obtain

$$\begin{aligned}
&(1/h^2)(\tfrac{1}{4}\delta_s^2\hat{u}_{i,j+1} + \tfrac{1}{2}\delta_s^2\hat{u}_{i,j} + \tfrac{1}{4}\delta_s^2\hat{u}_{i,j-1}) - (1/\omega_1^2k^2)\delta_t^2\hat{u}_{i,j} \\
&\quad + \mu\beta_2^2[(1/\omega_2^2k^2)\delta_t^2\hat{w}_{i,j} + (2/\omega_2hk)\delta_{st}^2\hat{w}_{i,j} + (1/k^2)(\tfrac{1}{4}\delta_s^2\hat{w}_{i,j+1} + \tfrac{1}{2}\delta_s^2\hat{w}_{i,j} \\
&\quad + \tfrac{1}{4}\delta_s^2\hat{w}_{i,j-1})]\,\delta(\hat{s} - \hat{\zeta}) = -\mu\beta_7 + N_1,
\end{aligned} \tag{16}$$

$$\begin{aligned}
&(1/h^4)(\tfrac{1}{4}\delta_s^4\hat{w}_{i,j+1} + \tfrac{1}{2}\delta_s^4\hat{w}_{i,j+1} + \tfrac{1}{4}\delta_s^4\hat{w}_{i,j-1}) - (1/\beta_5\omega_3^2)/(1/k^2)\delta_t^2\hat{w}_{i,j} - 2\beta_2^2[(1/\omega_2^2k^2)\delta_t^2\hat{w}_{i,j} \\
&\quad + (2/\omega_2hk)\delta_{st}^2\hat{w}_{i,j} + (1/k^2)(\tfrac{1}{4}\delta_s^2\hat{w}_{i,j+1} + \tfrac{1}{2}\delta_s^2\hat{w}_{i,j} + \tfrac{1}{4}\delta_s^2\hat{w}_{i,j-1})]\delta(\hat{s} - \hat{\zeta}) \\
&\quad + \mu\beta_2^2[(1/\omega_2^2k^2)\delta_t^2\hat{u}_{i,j} + (2/\omega_2hk)\delta_{st}^2\hat{u}_{i,j} + (1/k^2)(\tfrac{1}{4}\delta_s^2\hat{u}_{i,j+1} + \tfrac{1}{2}\delta_s^2\hat{u}_{i,j} \\
&\quad + \tfrac{1}{4}\delta_s^2\hat{u}_{i,j-1})]\,\delta(\hat{s} - \hat{\zeta}) = \beta_6 + N_2,
\end{aligned} \tag{17}$$

where $h = \Delta s$, $k = \Delta t$ and

$$\delta_s^4\hat{w}_{i,j+1} = \hat{w}_{i+2,j+1} - 4\hat{w}_{i+1,j+1} + 6\hat{w}_{i,j+1} - 4\hat{w}_{i-1,j+1} + \hat{w}_{i-2,j+1},$$

$$\delta_s^2\hat{w}_{i,j+1} = \hat{w}_{i+1,j+1} - 2\hat{w}_{i,j+1} + \hat{w}_{i-1,j+1}, \qquad \delta_t^2\hat{w}_{i,j} = \hat{w}_{i,j+1} - 2\hat{w}_{i,j} + \hat{w}_{i,j-1}, \quad \text{etc.}$$

N_1 and N_2 represent the sum of the non-linear terms in equations (14) and (15). Equations (16) and (17) can be easily written in matrix form. Neglecting non-linear terms, the solution to the linear systems are obtained. Then the non-linear terms can be approximated; for example

$$w_{ss}^2 = [(1/h^2)(\hat{w}_{i+1,j+1} - 2\hat{w}_{i,j+1} + \hat{w}_{i-1,j+1})]^2,$$

$$\hat{u}_s\hat{w}_{ss}^2 = (1/2h)(\hat{u}_{i+1,j+1} - \hat{u}_{i-1,j+1})[(1/h^2)(\hat{w}_{i+1,j+1} - 2\hat{w}_{i,j+1} + \hat{w}_{i-1,j+1})]^2, \quad \text{etc.}$$

Substituting the non-linear terms back into equations (16) and (17) yields a new system to be solved. This procedure was repeated until the desired convergence was achieved. Note that after equations (14) and (15) are linearized the two equations are coupled when friction is present, otherwise they uncoupled.

5. NUMERICAL SOLUTIONS

Traditionally, two basic mathematical models are considered when simulating a vehicle traversing a beam; the moving force problem and the moving mass problem [5, 6]. While the solutions presented are non-dimensional, they are clearly influenced by the stiffness and density of the bridge material, hence the following computations are for a steel structure.

Figure 4 is a plot of the longitudinal displacement at the moveable end of the beam, from which it can be seen that the longitudinal displacement depends mainly on the

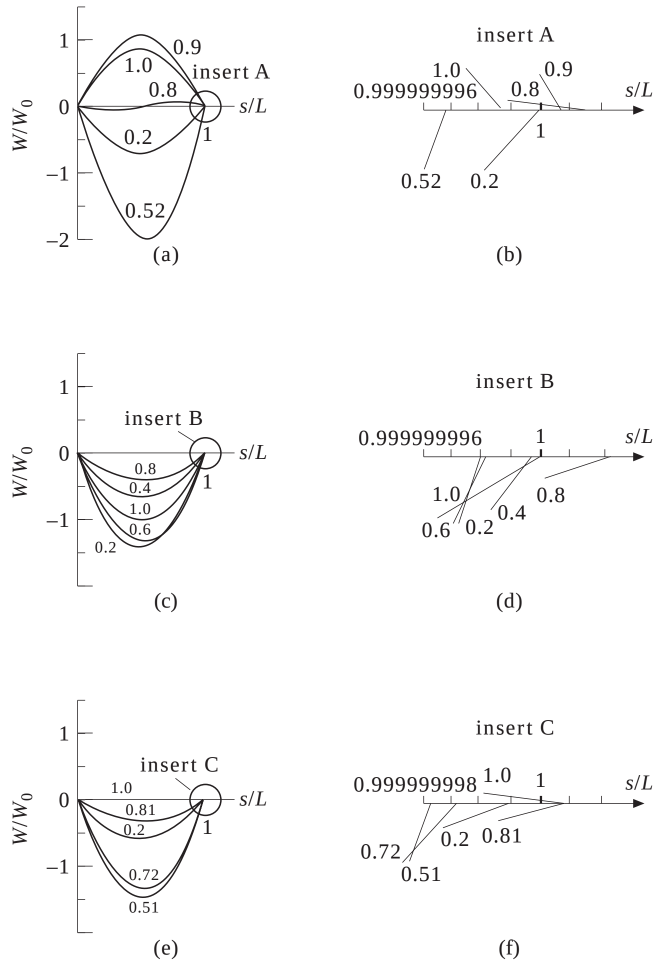

Figure 4. Time histories of the transverse and longitudinal displacement of the beam for different load velocities. Mass of vehicle/mass = 0·6. The numbering on each curve represents the point (percent of unity) that the vehicle is at on the beam when the deflection is recorded. (a) Transverse displacements for a vehicle traveling at 288 km/hr; (b) longitudinal displacements at the free end; (c) transverse displacements for a vehicle traveling at 130 km/hr; (d) longitudinal displacements at the free end; (e) transverse displacements for a vehicle traveling at 63.5 km/hr; (b) longitudinal displacements at the free end.

magnitude of the transverse displacement, but when the moving mass approaches the moveable end (within 20 percent of the bridge length) the axial force (the friction force) causes the beam to elongate. This is so because the elongation is a function of the length of the bridge behind the moving mass.

Figure 5 shows the difference in the two models and it can be seen that the moving mass approximation leads to the largest displacement and a higher frequency. Figure 6 shows the effect of the mass ratio (mass of vehicle/mass of bridge). As the figure clearly shows, the frequency of a loaded beam varies with the different mass ratios [7].

6. CONCLUDING REMARKS

The longitudinal and transverse motions of a finite elastic beam traversed by a moving mass has been studied. Using Hamilton's principle, two non-linear coupled

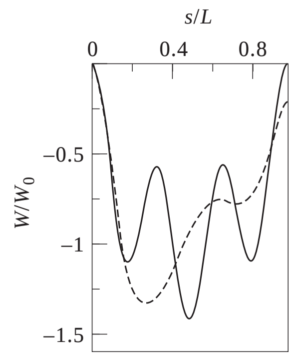

Figure 5. Time variation of the deflection at the center of the beam for the moving force and moving mass problems. Mass of the vehicle/mass of bridge = 1·2; v = 130 km/hr; w/w_0 = dynamic displacement/static displacement at the center of the beam.

differential equations governing the transverse and longitudinal displacements were developed. A finite different method combined with a perturbation technique was used to solve the resulting boundary value problem. The results show the differences between the moving force and moving mass problems for the dynamic system. The moving mass approximation leads to the largest displacement and a higher frequency. The effect of the friction force on the longitudinal motion was shown to be significant. The frequency of a loaded beam varies with the mass ratio, and the higher mass ratio leads to a lower frequency of vibration.

The authors' basic curiosity for pursuing this problem was to see what effect, if any, would result when one investigated the coupling between longitudinal and transverse motion produced by a mass moving over a beam. While the results are surprising, within elastic limits they are not significantly different than results obtained for pure bending without friction [5, 6]. On the other hand they are of significance when one considers alternate uses for the results. For example, consider a cutting tool traversing its guideway(bridge) when performing ultra precision cutting. Programs for controlling the cutting tool can be enhanced with the foregoing analysis.

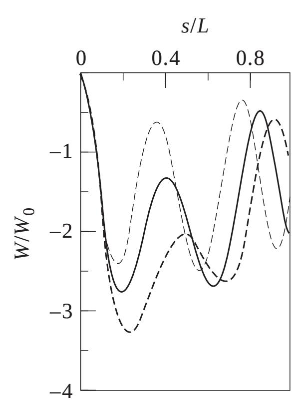

Figure 6. Time variation of the deflection at the center of the beam for several mass ratios (mass of vehicle/mass of bridge) when v = 130 km/hr. Mass ratios: ----, 0·3; ——, 0·6; ----, 0·9.

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APPENDIX: LIST OF SYMBOLS

A cross-sectional area of the beam
EI bending rigidity of the beam
F_x external concentrated force in horizontal direction
F_y external concentrated force in vertical direction
g gravitational acceleration
L length of the beam
m mass per unit length of the beam
M external concentrated moment caused by the moving mass
R_i reaction forces at the ends of the beam
s spatial co-ordinate of the beam
T kinetic energy of the beam
u longitudinal deflection of the beam
U work done by the external loads
v velocity of the moving mass
V total potential energy of the deformed beam
V_1 potential energy caused by bending
V_2 potential energy caused by extension
w transverse deflection of the beam
W weight of the moving mass
ε_x longitudinal strain
ζ moving mass position
θ angle of inclination
μ friction coefficient
σ_x longitudinal stress